

POLAR ACTIONS ON SYMMETRIC SPACES OF HIGHER RANK

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ABSTRACT. We show that polar actions of cohomogeneity two on simple compact Lie groups of higher rank, endowed with a biinvariant Riemannian metric, are hyperpolar. Combining this with a recent result of the second-named author, we are able to prove that polar actions (of arbitrary cohomogeneity) induced by reductive algebraic subgroups in the isometry group of an irreducible Riemannian symmetric space of higher rank (compact or non-compact) are hyperpolar.

1. INTRODUCTION

An isometric action of a compact Lie group on a Riemannian manifold is called *polar* if there exists an immersed submanifold which meets every orbit such that the orbits intersect the submanifold orthogonally at each of its points. Such a submanifold is called a *section* of the Lie group action. If there is a section which is flat in its induced Riemannian metric, then the action is called *hyperpolar*. The main result of this article is the following.

Theorem 1. *Polar actions with orbits of positive dimension on irreducible compact symmetric spaces of rank greater than one are hyperpolar.*

Theorem 1 has been proved already in various special cases. Podestà and Thorbergsson have shown that polar actions on the quadric $\mathrm{SO}(n+2)/\mathrm{SO}(n)\mathrm{SO}(2)$ are hyperpolar, using the fact that polar actions on this space are coisotropic, see [16]. This result has been generalized by Biliotti and Gori [2], [3] to all compact irreducible Hermitian symmetric spaces, leading Biliotti to formulate the statement of Theorem 1 as a conjecture. In the special cases of symmetric spaces with simple compact isometry group and of exceptional compact Lie groups with a biinvariant metric, Theorem 1 has been proved by the first-named author [10], [11]. Recently, the second-named author has shown that polar singular Riemannian foliations of

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irreducible compact symmetric spaces of higher rank are hyperpolar if the codimension of the foliation is at least three [13]. To prove Theorem 1 it therefore suffices to show that a polar cohomogeneity two action on a (classical) compact Lie group endowed with a biinvariant Riemannian metric is hyperpolar.

Note that Theorem 1 does not generalize to non-compact symmetric spaces. In fact, there are counterexamples of homogeneous polar foliations (i.e. polar actions all of whose orbits are principal) with non-flat sections, see [1, Proposition 4.2]. However, Theorem 1 still holds for actions on non-compact irreducible spaces if one requires the action to be given by a reductive algebraic subgroup of the isometry group. (For example, semisimple subgroups and compact subgroups of a semisimple Lie group are reductive algebraic subgroups, see [15].)

Corollary 1. *Let M be an irreducible symmetric space. Let H be a reductive algebraic subgroup of the isometry group of M . If the action of H on M is polar and has orbits of positive dimension, then the action is hyperpolar or M is of rank one.*

Corollary 1 follows immediately from Theorem 1 and [12, Theorem 5.1]. Note that it was shown in [12] that hyperpolar actions of reductive algebraic subgroups in the isometry group of irreducible symmetric spaces are of cohomogeneity one or Hermann actions.

2. PRELIMINARIES

Let L be a compact connected simple Lie group equipped with the biinvariant Riemannian metric induced by the negative of the Killing form. Then L is a Riemannian symmetric space [7]. The connected component of the isometry group is covered by $L \times L$ and to study polar actions on L it suffices to consider closed connected subgroups of $L \times L$. Henceforth we assume L is a simple compact Lie group of rank greater than one and $H \subset L \times L$ is a closed connected subgroup which acts polarly, but not hyperpolarly, and with cohomogeneity two on L by its natural action, which is given by $(h_1, h_2) \cdot \ell := h_1 \ell h_2^{-1}$ for all $(h_1, h_2) \in H$ and all $\ell \in L$. We will prove that no such subgroup exists.

Each connected closed subgroup of $L \times L$ acting nontransitively on L is contained in (at least) one of the following subgroups of $L \times L$, see Dynkin [4, Theorem 15.1]: In a diagonal subgroup of the form

$$(2.1) \quad \Delta^\sigma L := \{(\ell, \sigma(\ell)) \mid \ell \in L\}$$

where σ is an automorphism of L , or in a subgroup of the form

$$(2.2) \quad H_1 \times H_2 := \{(h_1, h_2) \mid h_1 \in H_1, h_2 \in H_2\},$$

where H_1 and H_2 are closed connected proper subgroups of L .

Lemma 2.1. *If a closed connected subgroup H of $\Delta^\sigma L$ acts polarly on the simple compact connected Lie group L of rank greater than one, then $H = \Delta^\sigma L$ or $H = \{e\}$.*

Proof. [11, Proposition 12]. □

The action of $\Delta^\sigma L$ on L is well-known to be hyperpolar; these actions were named σ -actions in [6]. Hence it suffices to consider those subgroups of $L \times L$ which are contained in groups of the form (2.2).

The following criterion for polarity of isometric actions on symmetric spaces is well-known, for a proof see for instance [10, Proposition 4.1]. By $H \cdot eK$ we denote the H -orbit through the point $eK \in G/K$ and by $N_{eK}(H \cdot eK)$ its normal space at the point eK .

Proposition 2.2. *Let G be a connected semisimple compact Lie group, let σ be an involutive automorphism of G and let $K \subset G$ be a closed subgroup fulfilling $G_0^\sigma \subseteq K \subseteq G^\sigma$ such that G/K is an irreducible symmetric space. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition into eigenspaces of $d\sigma_e$. Let $H \subset G$ be a closed subgroup. Assume the element eK of G/K lies in a principal orbit of the H -action on G/K . Then the following are equivalent.*

- (i) *The H -action on G/K is polar w.r.t any G -invariant Riemannian metric on G/K .*
- (ii) *The subspace $\nu = N_{eK}(H \cdot eK) \subseteq \mathfrak{p}$ is a Lie triple system such that the Lie algebra $\nu \oplus [\nu, \nu]$ generated by ν is orthogonal to \mathfrak{h} w.r.t. to the negative of the Killing form on \mathfrak{g} .*

In particular, the polarity of an action may be decided on the Lie algebra level. The Lie triple system ν which appears in the proposition corresponds to the tangent space of a section in case of a polar action and so the H -action is hyperpolar if and only if ν is abelian. In the special case of a compact Lie group L endowed with a biinvariant Riemannian metric, viewed as a Riemannian symmetric space, we use $G = L \times L$, $K = \Delta L := \{(\ell, \ell) \mid \ell \in L\}$ and $\mathfrak{p} = \{(X, -X) \mid X \in \mathfrak{g}\}$.

3. POLAR COHOMOGENEITY TWO ACTIONS ON COMPACT LIE GROUPS

The following observation will be useful.

Lemma 3.1. *Let L be a simple compact Lie group and let $H_1, H_2 \subset L$ be closed subgroups. Assume the group*

$$H_1 \times H_2 = \{(h_1, h_2) \mid h_1 \in H_1, h_2 \in H_2\}$$

acts polarly and with cohomogeneity two on L . Then the $H_1 \times H_2$ -action on L is hyperpolar.

| H_1 | L | H_2 |
|---------------------|-------------------|---|
| $\mathrm{Sp}(n)$ | $\mathrm{SU}(2n)$ | $\mathrm{S}(\mathrm{U}(2n-1) \mathrm{U}(1))$ $\mathrm{SU}(2n-1)$ |
| $\mathrm{SO}(2n-1)$ | $\mathrm{SO}(2n)$ | $\mathrm{U}(n)$ $\mathrm{SU}(n)$ |
| $\mathrm{SO}(4n-1)$ | $\mathrm{SO}(4n)$ | $\mathrm{Sp}(n) \cdot \mathrm{Sp}(1)$ $\mathrm{Sp}(n) \cdot \mathrm{U}(1)$ $\mathrm{Sp}(n)$ |
| G_2 | $\mathrm{SO}(7)$ | $\mathrm{SO}(6)$ |
| G_2 | $\mathrm{SO}(7)$ | $\mathrm{SO}(5) \mathrm{SO}(2)$ $\mathrm{SO}(5)$ |
| $\mathrm{Spin}(7)$ | $\mathrm{SO}(8)$ | $\mathrm{SO}(7)$ |
| $\mathrm{Spin}(9)$ | $\mathrm{SO}(16)$ | $\mathrm{SO}(15)$ |

TABLE 1. Transitive actions.

Proof. Assume the action of $H_1 \times H_2 \subset L \times L$ on L is polar, but not hyperpolar and let $\mathfrak{l} = \mathfrak{h}_1 + \mathfrak{m}_1 = \mathfrak{h}_2 + \mathfrak{m}_2$ be orthogonal decompositions of the Lie algebra of L . We may assume that the identity element $e \in L$ lies in a principal orbit. By Proposition 2.2 (ii), there are two noncommuting vectors $(X, -X), (Y, -Y) \in \nu$ such that $(Z, Z) \perp \mathfrak{h}$, where $Z = [X, Y]$. Since the Lie algebra of L does not contain a two-dimensional nonabelian subalgebra, we have that $X, Y, Z \in \mathfrak{m}_1 \cap \mathfrak{m}_2$ are linearly independent, contradicting the assumption that $H_1 \times H_2$ acts on L with cohomogeneity two. \square

We can now prove the following lemma, which shows that cohomogeneity two polar actions with non-flat sections on simple compact Lie groups of higher rank can only occur as restrictions of cohomogeneity one actions.

Lemma 3.2. *Let L be a simple compact Lie group of rank greater than one and assume $H \subset L \times L$ is a closed connected subgroup whose action on L is polar and of cohomogeneity two, but not hyperpolar. Then there are closed connected proper subgroups H_1 and H_2 of L such that H is contained in $H_1 \times H_2$ and such that the action of $H_1 \times H_2$ on L given by $(h_1, h_2) \cdot \ell := h_1 \ell h_2^{-1}$ is of cohomogeneity one.*

Proof. Define H_1 and H_2 to be the images of H under the natural projections onto the first and second factor of $L \times L$, respectively. Then H_1 and H_2 are compact connected subgroups of L and H is contained in $H_1 \times H_2$. Define the ideals $\mathfrak{h}'_1 := \ker(\pi_2|_{\mathfrak{h}})$ and $\mathfrak{h}'_2 := \ker(\pi_1|_{\mathfrak{h}})$ of \mathfrak{h} , where the maps π_1 and π_2 are the natural projections onto the first and second factor of the Lie algebra of $L \times L$. Let \mathfrak{h}_Δ

be an ideal which is complementary to $\mathfrak{h}'_1 + \mathfrak{h}'_2$ in \mathfrak{h} . The maps $\pi_1|_{\mathfrak{h}_\Delta + \mathfrak{h}'_1}$ and $\pi_2|_{\mathfrak{h}_\Delta + \mathfrak{h}'_2}$ are injective and we have $\mathfrak{h}_1 = \pi_1(\mathfrak{h}_\Delta) \oplus \mathfrak{h}'_1$ and $\mathfrak{h}_2 = \pi_2(\mathfrak{h}_\Delta) \oplus \mathfrak{h}'_2$. It follows from the nontransitivity of the H -action on L and Lemma 2.1 that H_1 and H_2 are proper subgroups of L . Since the $H_1 \times H_2$ -action on L is not orbit equivalent to the H -action on L by Lemma 3.1, it follows that \mathfrak{h}_Δ is a nontrivial Lie algebra and that $H_1 \times H_2$ acts either with cohomogeneity one or transitively on L . All transitive actions of groups $H_1 \times H_2$ on simple compact Lie groups L were determined by Oniščik [14], the result is given by Table 1. Inspection of Table 1 shows that there is only one case in which H_1 and H_2 have an isomorphic ideal, namely the case of the $\text{Spin}(7) \times \text{SO}(7)$ -action on $\text{SO}(8)$. However, in this case $\mathfrak{h} = \mathfrak{h}_\Delta$ would be isomorphic to the 21-dimensional Lie algebra $\mathfrak{so}(7)$, contradicting the assumption that H acts on the 28-dimensional Lie group $\text{SO}(8)$ with cohomogeneity two. It follows that the $H_1 \times H_2$ -action on L is of cohomogeneity one. \square

Remark 3.3. It should be noted that Table 1 has to be interpreted on the Lie algebra level. Indeed, the transitive actions on $\text{SO}(8)$ of $\text{Spin}(7) \times [\text{SO}(6)\text{SO}(2)]$ and of $\text{Spin}(7) \times [\text{SO}(5)\text{SO}(3)]$ correspond to the actions of $\text{SO}(7) \times \text{U}(4)$ and $\text{SO}(7) \times [\text{Sp}(2) \cdot \text{Sp}(1)]$, respectively, via a triality automorphism of $\text{Spin}(8)$, cf. [9, Proposition 3.3].

4. SUBACTIONS OF COHOMOGENEITY ONE ACTIONS

In this section we are going to prove the following result that finishes the proof of the main theorem.

Proposition 4.1. *Let M be a compact irreducible symmetric space of higher rank with a polar and not hyperpolar action of a group H of cohomogeneity two. Then H is not a subgroup of a larger group H' that acts on M with cohomogeneity one.*

Proof. Assume that such a group H' does exist. We may assume that H and H' are connected. Denote by Δ the quotient space M/H . By polarity of the action of H , Δ is a finite quotient of a section Σ , which is a two-dimensional symmetric space. By assumption Σ is non-flat. Thus it is either the round sphere or the round projective space, which, after rescaling, may be assumed to be of constant curvature 1. In any case, Δ is the quotient of the universal covering S^2 of Σ by a finite Coxeter group W .

In [13, Section 7] it is shown that if the Coxeter group W is reducible, the space M must be of rank one. Thus we may assume that W is irreducible. Then Δ is a spherical triangle with angles $\pi/2, \pi/3, \pi/m$, with $m = 3$ or $m = 4$. (In fact, the case $m = 3$, can be excluded using the ideas of [13], cf. [5], but this will not be used in the sequel). Hence the triangle Δ has two vertices x, y such that the angles at these points are equal to π/m with (possibly different) $m > 2$.

Let $p \in M$ be any point in the H -orbit corresponding to x . Then the isotropy group H_p acts on the normal space V to the orbit $H \cdot p$ with cohomogeneity two, such that the quotient V/H_p is the tangent space to Δ at x , i.e., the cone over the interval of length π/m . Hence the action of H_p on V is irreducible. Therefore, the larger isotropy group H'_p of H' at p cannot act on a proper non-trivial subspace of V . Therefore, the normal space to the orbit $H' \cdot p$ at p coincides with V . Thus the orbits $H' \cdot p$ and $H \cdot p$ coincide.

The same argument works for a point q over the vertex y . Thus we deduce that the orbits of H and of H' through p and q coincide. Therefore, $H \cdot p$ and $H \cdot q$ are the only singular orbits of the cohomogeneity one action of H' on M .

Take any regular point o in the section Σ from above. Then this point o must lie on a shortest H' -horizontal geodesic from $H \cdot p$ to $H \cdot q$. This geodesic is also H -horizontal, hence contained in Σ . For a generic choice of o in Σ , we derive a contradiction. \square

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